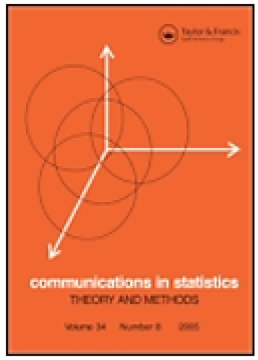

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Mixture of the Riesz distribution with respect to a multivariate Poisson

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Abstract The aim of this paper is to study a statistical model obtained by the mixture of the Riesz probability distribution on symmetric matrices with respect to a multivariate Poisson distribution. We show that this distribution is related to the modified Bessel function of the first kind. We then determine the domain of the means and the variance function of the generated natural exponential family.

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1 Introduction

The mixture of distributions has provided a mathematical-based approach to the statistical modeling of a wide variety of random phenomena. It is an extremely flexible method of modeling with a wide range of applicability in nearly all areas. Indeed, the extent of the application of the mixture of distributions includes psychology (Bromet *et al.* (1985)), biology (Stigler, (1986)), genetics (Liang and Rathouz (1999)), ecology (Davis *et al.* (2004)) and medicine (Foll *et al.* (2005) or Liu *et al.* (2007)). It also includes many fields in engineering (Harris and Singpurwalla (1968)), and in social sciences (Formann and Kohlmann (1996)). This has drawn considerable interest to the statistical inference for mixture distributions and has involved interesting estimation problems. In this context, Hill, Laud and Saunders (1980) have worked on maximum likelihood estimation of the mixing probabilities and Laird (1978) has obtained some general results about the existence of maximum likelihood estimates of arbitrary mixing measures. The progress of using the mixture of distributions has been particulary evident in the Bayesian approach, where it began with the Gibbs sampling algorithm of Diebolt and Robert (1994) for estimating the parameters of a mixture with a fixed number of components. We also mention that the approach based on mixing distributions provides a smooth estimate of the survival distribution function which may be preferable to the standard nonparametric step function estimates. In the simplest case the mixture consists in considering weighted linear combination of some underlying basis distributions. The mixing parameter may also be

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denumerably infinite, as in the theory of sums of a random number of random variables, or continuous, as in the compound Poisson distribution. In these examples and in almost all the works, the mixing parameter is the convolution power which is either an integer or a positive real number, and consequently the mixing distribution is concentrated on $\mathbb{R}$. In the present work, we deal with a very special case in which the model is defined on the cone Ω of (r, r) positive definite symmetric matrices and the mixing parameter belongs to a subset of $\mathbb{R}^r$. More precisely, the model is in the absolutely continuous Riesz model

$$\left\{R(s,\sigma),\ s \in \prod_{i=1}^{r} \left]\frac{i-1}{2}, +\infty\right[\right\},$$

introduced in Hassairi and Lajmi (2001). Here the scale parameter σ is in Ω, and the shape parameter $s = (s_1, s_2, \ldots, s_r)$ may be viewed as a multivariate parameter of convolution, since we have the property

$$R(s,\sigma) * R(s',\sigma) = R(s+s',\sigma),$$

which is analogous to the one satisfied by the ordinary powers of convolution. The parameter s will be used as a mixing parameter, and the mixing distribution will be a multivariate Poisson distribution on $\mathbb{N}^r$. We mention that the Riesz model contains the Wishart distributions since $R(s,\sigma)$ reduces to a Wishart distribution when $s_1 = s_2 = \ldots = s_r = p$, and in this case, p is nothing but the ordinary convolution parameter. It is well known that the Wishart distribution plays a prominent role in the estimation of covariance matrices in multivariate statistics, it is also of particular importance in Bayesian inference, as it is the conjugate prior of the inverse of the covariance matrix of a multivariate normal distribution. We haven't yet a statistical interpretation of the Riesz model, in its general form, nevertheless several interesting results concerning this distribution and some other related distribution (see Hassairi *et al.* (2005), Hassairi *et al.* (2007), Ben Farah and Hassairi (2007) or Hassairi *et al.* (2009)). The definition of a Riesz distribution has also been extended to homogeneous cones in connection with graphical models (see Boutouria (2009) or Andersson and Klein (2010)). Accordingly, there is a need for more thorough investigation of theoretical and applied aspects of the Riesz distribution which may lead to some statistical applications. The paper's main result is within this framework, it extends the approach based on the mixture of distributions to the Riesz dispersion model on symmetric matrices. This mixture has a theoretical interest due to the nature of the model and of the mixing parameter. It has also a particular importance due to the role of symmetric positive definite matrices in the multivariate statistical analysis. In particular, it provides a rich class of matrix-variate natural exponential families, and consequently, a more flexible method of modeling on symmetric matrices. It also poses open estimation problems more general than the ones related to the Wishart distribution. The paper is organized as follows: In Section 2, we recall some definitions and give some preliminary results relevant to the mixture of distributions. In Section 3, we introduce the Riesz exponential dispersion models on symmetric matrices. In Section 4, we state and prove our main results concerning the mixture of the Riesz distribution with respect to a multivariate Poisson. We show that the mixture distribution is expressed in terms of the modified Bessel function. We then determine the domain of the means of the generated natural exponential family, and we calculate its variance function.

2 The notion of mixture

Let μ_λ be a probability distribution on a finite dimensional linear space E depending on a parameter λ which belongs to a subset Λ of $\mathbb{R}^r$. Suppose that

$$\mu_\lambda = f(x,\lambda)\sigma(dx),$$

where σ is some reference measure, and that for each x in E, the map $\lambda \mapsto f(x,\lambda)$ defined on Λ is measurable. For a probability distribution $\nu(d\lambda)$ on the set Λ, define

$$h(x) = \int_\Lambda f(x,\lambda)\nu(d\lambda).$$

Then the probability measure

$$\mu_\nu(dx) = h(x)\sigma(dx)$$

is called the mixture of the distribution μ_λ with respect to ν. (See Feller (1971), Vol. II, page 53 or Johnson *et al.* (2005), page 360). Usually, ν is called the mixing distribution (see Karlis and Meligkotsidou (2007)).

A special case of interest is when μ_λ is the $\lambda-$power of convolution of a measure μ which is not concentrated on an affine hyperplane of E, and Λ is the so called Jørgensen set of μ. Specifically, let

$$L_\mu(\theta) = \int_E \exp(\langle\theta, x\rangle)\mu(dx) \tag{2.1}$$

denote the Laplace transform of μ in $\theta \in E^*$, where $\langle,\rangle$ is the duality bracket, and suppose that the set

$$\Theta(\mu) = \text{interior}\{\theta \in E^*;\ L_\mu(\theta) < +\infty\} \tag{2.2}$$

is nonempty. Then the set

$$\Lambda = \{\lambda > 0;\ \exists\ \ \mu_\lambda \text{ such that } L_{\mu_\lambda}(\theta) = (L_\mu(\theta))^\lambda,\ \text{ for all } \theta \in \Theta(\mu)\} \tag{2.3}$$

is called the Jørgensen set of μ and the measure μ_λ is its $\lambda-$power of convolution. Of course, for λ and λ' in Λ, we have that $\mu_\lambda * \mu_{\lambda'} = \mu_{\lambda+\lambda'}$. The set Λ is equal to $]0,+\infty[$ if and only if μ is infinitely divisible (see Seshadri (1994), page 155). It contains always the set $\mathbb{N}^*$ of positive integers, so that for any distribution μ and any positive integer N, one may consider the distribution μ_N as defined in (2.3). When μ is discrete, i.e., with countable support, the mixture of μ with respect to a distribution ν on the parameter N is known as a compound distribution. The most famous compound distribution is the one corresponding to the case where ν is Poisson (see Feller (1971), Vol. I, page 286 or Vol. II, page 451 or Aalen (1992) or Jørgensen (1997), page 140). In fact, the real Poisson distributions appear in numerous works either as elements of the model (see Johnson *et al.* (2005), page 366) or as mixing distributions (see Perline (1988)).

3 The Riesz exponential dispersion model

In this section, we recall some general facts concerning the exponential dispersion models in an Euclidean space, and we introduce the Riesz model on symmetric matrices.

3.1 Exponential dispersion model

Let E be an Euclidean space with finite dimension n, and let $\langle , \rangle$ denote the scalar product in E. If μ is a positive measure on E, we denote by $\mathcal{M}(E)$ the set of measures μ such that $\Theta(\mu)$ defined by (2.2) is not empty and μ is not concentrated on an affine hyperplane. The cumulant function of an element μ of $\mathcal{M}(E)$ is the function defined for θ in $\Theta(\mu)$ by

$$k_\mu(\theta) = \log L_\mu(\theta),$$

where L_μ is the Laplace transform of μ defined in (2.1).

To each μ in $\mathcal{M}(E)$ and θ in $\Theta(\mu)$, we associate the probability distribution on E

$$P(\theta,\mu)(dx) = exp\left(\langle\theta, x\rangle - k_\mu(\theta)\right)\mu(dx).$$

The set

$$F = F(\mu) = \{P(\theta,\mu);\ \theta \in \Theta(\mu)\}$$

is called the natural exponential family (NEF) generated by μ. The first derivative k'_μ of k_μ defines a diffeomorphism between $\Theta(\mu)$ and its image M_F called the domain of the means of F. The inverse function of k'_μ is denoted by ψ_μ and setting $P(m, F) = P(\psi_\mu(m), \mu)$ the element of F with mean m, we have $F = \{P(m, F);\ m \in M_F\}$, which is the parametrization of F by the mean.

Now the covariance operator of $P(m, F)$ is denoted by $V_F(m)$ and the map

$$M_F \longrightarrow L_s(E);\ m \longmapsto V_F(m) = k''_\mu(\psi_\mu(m)) = (\psi'_\mu(m))^{-1}$$

is called the variance function of F. An important feature of V_F is that it characterizes F in the following sense: If F and F' are two NEFs such that $V_F(m)$ and $V_{F'}(m)$ coincide on a nonempty open subset of $M_F \cap M_{F'}$, then $F = F'$.

If μ is an element of $\mathcal{M}(E)$ and Λ is its Jørgensen set defined by (2.3). Then the set

$$\{P(\theta,\lambda) = \exp\left(\langle\theta, x\rangle - k_{\mu_\lambda}(\theta)\right)\mu;\ \theta \in \theta(\mu),\ \lambda \in \Lambda\}$$

is called the dispersion model generated by μ. For more details, we refer to Letac (1992).

3.2 Riesz natural exponential families

Let E be the Euclidean space of (r, r) real symmetric matrices equipped with the scalar product $\langle x, y\rangle = \mathrm{tr}(xy)$, and the inner product $x.y = \frac{1}{2}\Big(xy + yx\Big)$, where xy is the ordinary product of two matrices. We denote by $e_1, e_2, \ldots, e_r$ the canonical basis of $\mathbb{R}^r$; $e_i = (0, \ldots, 0, 1, 0 \ldots 0)$, (1 in the i$^{\text{th}}$ place), and we set $c_i = diag(e_i)$ for all $1 \leq i \leq r$.

For $x \in E$, we consider the endomorphism of E defined by $L(x):\ y \longmapsto x.y$, and we set

$$P(x) = 2(L(x))^2 - L(x^2). \tag{3.4}$$

For $x = (x_{ij})_{1\leq i,j\leq r}$ in E and $1 \leq k \leq r$, we define the sub-matrices

$$P_k(x) = (x_{ij})_{1\leq i,j\leq k} \text{ and } P^*_k(x) = (x_{ij})_{r-k+1\leq i,j\leq r}.$$

For convenience we may view the matrices $P_k(x)$ and $P^*_k(x)$ as elements of the space E, by suitably augmenting them with rows and columns of zeros. We set $P^*_0(x) = 0$.

Let $\Delta_k(x)$ and $\Delta_k^*(x)$ denote the determinant of the (k,k) matrix $P_k(x)$ and the determinant of the (k,k) matrix $P_k^*(x)$, respectively. Then the generalized power of x in the cone Ω of positive definite elements of E is defined, for $s=(s_1,s_2,\dots,s_r)\in\mathbb{R}^r$, by

$$\Delta_s(x)=\Delta_1(x)^{s_1-s_2}\Delta_2(x)^{s_2-s_3}\dots\Delta_{r-1}(x)^{s_{r-1}-s_r}\Delta_r(x)^{s_r}. \tag{3.5}$$

Note that if for all $i\in\{1,\dots,r\}$, $s_i=p\in\mathbb{R}$, then $\Delta_s(x)=(\det x)^p$. We also define

$$\Delta_s^*(x)=(\Delta_1^*(x))^{s_1-s_2}(\Delta_2^*(x))^{s_2-s_3}\dots(\Delta_{r-1}^*(x))^{s_{r-1}-s_r}(\Delta_r^*(x))^{s_r}. \tag{3.6}$$

It is shown (see Hassairi and Lajmi (2001)) that for all $x\in\Omega$ and all $s\in\mathbb{R}^r$, we have

$$\Delta_s(x^{-1})=\Delta^*_{-s^*}(x), \tag{3.7}$$

where $s^*=(s_r,s_{r-1},\dots,s_1)$.

We denote by $\mathcal{T}_l^+$ the set of lower triangular matrices with positive diagonal elements. For $u\in\mathcal{T}_l^+$, we denote by u^* the transpose matrix of u, and we define on E the automorphism

$$u(y)=uyu^*, \tag{3.8}$$

It is well known that for all $x\in\Omega$, there exists a unique $u\in\mathcal{T}_l^+$ such that $x=u(I_r)$, where I_r is the identity matrix of order r, this is the Cholesky decomposition of x.

We also have (see Hassairi and Lajmi (2001)) that for all $1\le i\le r$,

$$\left(P_i^*\left((u(I_r))^{-1}\right)\right)^{-1}=u\left(\sum_{k=r-i+1}^{r}c_k\right), \tag{3.9}$$

and for all $s=(s_1,s_2,\dots,s_r)\in\mathbb{R}^r$,

$$\Delta_s(u(I_r))=\Delta^*_{-s^*}(u^{*-1}(I_r)). \tag{3.10}$$

Recall that for $x\in\Omega$ and $u\in\mathcal{T}_l^+$, we have (See Faraut and Korányi (1994), page 114),

$$\Delta_i(u(x))=\Delta_i(u(I_r))\Delta_i(x)=u_1^2\dots u_i^2\Delta_i(x), \tag{3.11}$$

where for all $i\in\{1,\dots,r\}$,

$$u_i=\langle u,c_i\rangle. \tag{3.12}$$

Consider for $s=(s_1,s_2,\dots,s_r)\in\prod_{i=1}^{r}\left]\frac{i-1}{2},+\infty\right[$, the absolutely continuous Riesz measure

$$R_s(dx)=\frac{\Delta_{s-\frac{n}{r}}(x)}{\Gamma_\Omega(s)}\mathbf{1}_\Omega(x)(dx),$$

where $n=\dfrac{r(r+1)}{2}$ is the dimension of E and

$$\Gamma_\Omega(s)=(2\pi)^{\frac{r(r-1)}{4}}\prod_{i=1}^{r}\Gamma\left(s_i-\frac{i-1}{2}\right). \tag{3.13}$$

Then a result due to Gindikin (1964) says that for all $\theta \in -\Omega$,

$$L_{R_s}(\theta) = \int_E e^{\langle \theta, x \rangle} R_s(dx) = \Delta_s(-\theta^{-1}).$$

The natural exponential family measure generated by R_s is

$$F = F(R_s) = \left\{ R(s,\sigma)(dx) = \frac{e^{-<\sigma,x>}\Delta_{s-\frac{n}{r}}(x)}{\Gamma_\Omega(s)\Delta_s(\sigma^{-1})} \mathbf{1}_\Omega(x)(dx),\ \sigma \in \Omega \right\}.$$

The distribution $R(s,\sigma)$ is called the Riesz distribution with parameters s and σ, its Laplace transform is given for θ in $\sigma - \Omega$ by

$$L_{R(s,\sigma)}(\theta) = \frac{\Delta_s((\sigma-\theta)^{-1})}{\Delta_s(\sigma^{-1})}. \tag{3.14}$$

This implies that if σ is an element of Ω and if s and s' are in $\prod_{i=1}^{r} \left]\frac{i-1}{2}, +\infty\right[$, then

$$R(s,\sigma) * R(s',\sigma) = R(s+s',\sigma).$$

When $s_1 = s_2 = \ldots = s_r = p > \frac{r-1}{2}$, $R(s,\sigma)$ reduces to the Wishart distribution

$$W(p,\sigma)(dx) = \frac{1}{\Gamma_\Omega(p)\det(\sigma^{-p})} e^{-<\sigma,x>} \det(x)^{p-\frac{n}{r}} \mathbf{1}_\Omega(x)(dx), \tag{3.15}$$

with Laplace transform $L_{W(p,\sigma)}(\theta) = \det\left(I_r - \sigma^{-1}\theta\right)^{-p}$. The domain of the means of the Riesz family $F = F(R_s)$ is Ω, its variance function is

$$V_F(m) = \sum_{i=1}^{r}(s_{r-i+1}-s_{r-i})P\left(\frac{1}{s_{r-i+1}}\left(P_i^*(m^{-1})\right)^{-1} + \sum_{k=1}^{i-1}\left(\frac{1}{s_{r-k+1}} - \frac{1}{s_{r-k}}\right)\left(P_k^*(m^{-1})\right)^{-1}\right), \tag{3.16}$$

(see Hassairi and Lajmi (2001)).
Note that the Riesz exponential NEF belongs to the Tweedie scale on symmetric matrices (see Hassairi and Louati (2009)), and that V_F is a rational fraction which reduces to a polynomial in the case where $s_1 = \ldots = s_r = p$, that is, in the case of a Wishart family. In this case (3.16) becomes

$$V_F(m) = \frac{P(m)}{p} \text{ for all } m \in \Omega.$$

4 The mixture of the Riesz distribution with respect to a multivariate Poisson

In this section, we state and prove our main results concerning the mixture of the Riesz distribution on symmetric matrices with respect to a multivariate Poisson. For simplicity, we will be interested in the case where $\sigma = I_r$, the identity matrix of size r.
Consider the Poisson distribution on $\mathbb{N}^r$ with parameter $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in]0,\infty[^r$

$$\nu(dx) = e^{-\sum_{i=1}^{r}\lambda_i} \sum_{q \in \mathbb{N}^r} \frac{\lambda^q}{q!}\ \delta_q(dx), \tag{4.17}$$

where $q! = q_1!\, q_2! \dots q_r!$ and $\lambda^q = \lambda_1^{q_1} \lambda_2^{q_2} \dots \lambda_r^{q_r}$. Then for all $\theta \in \mathbb{R}^r$,

$$L_\nu(\theta) = \prod_{i=1}^{r} e^{\lambda_i (e^{\theta_i} - 1)}. \tag{4.18}$$

Now let $\rho = \left(0, \frac{1}{2}, \dots, \frac{r-1}{2}\right)$, and for $k = (k_1, k_2, \dots, k_r)$ in $\mathbb{N}^r$, define

$$\widetilde{R}_k = R(k + \rho, I_r). \tag{4.19}$$

Suppose that $k = (k_1, k_2, \dots, k_r)$ has the multivariate Poisson distribution ν defined in (4.17). Denote by μ the mixture of $\widetilde{R}_k$ with respect to ν. The following theorem gives the expression of μ in terms of the modified Bessel function of the first kind.

Theorem 4.1

$$\mu(dx) = \frac{e^{-tr(x)}}{(2\pi)^{\frac{r(r-1)}{4}} \sqrt{\det(x)}} \prod_{i=1}^{r} \frac{\sqrt{\lambda_i}\; e^{-\lambda_i}}{\sqrt{\Delta_{i-1}(x)}}\, I\left(1, 2\sqrt{\lambda_i \Delta_{e_i}(x)}\right) \mathbf{1}_\Omega(x)(dx),$$

where $I(1,t)$ is the modified Bessel function of the first kind and of order 1 and $\Delta_0(x) = 1$.

Proof. Let $a = (a_1, a_2, \dots, a_r)$ be an element of $]0, +\infty[^r$ and let

$$\widetilde{R}_{k,a} = R(k + \rho + a, I_r).$$

Denote by μ_a the mixture of $\widetilde{R}_{k,a}$ with respect to ν. Then

$$\mu_a(dx) = h_a(x) \mathbf{1}_\Omega(x)(dx),$$

where

$$h_a(x) = e^{-\sum_{i=1}^{r} \lambda_i} \sum_{q \in \mathbb{N}^r} \frac{\lambda^q\; e^{-tr(x)} \Delta_{q+\rho+a-\frac{n}{r}}(x)}{q!\; \Gamma_\Omega(q + \rho + a)}. \tag{4.20}$$

Using (3.5), we can write

$$\Delta_{q+\rho+a-\frac{n}{r}}(x) = \Delta_1(x)^{q_1 - q_2 + a_1 - a_2 - \frac{1}{2}} \dots \Delta_{r-1}(x)^{q_{r-1} - q_r + a_{r-1} - a_r - \frac{1}{2}} \Delta_r(x)^{q_r + a_r - 1}.$$

$$= \left(\frac{\Delta_1(x)}{\Delta_0(x)}\right)^{q_1 + a_1} \left(\frac{\Delta_2(x)}{\Delta_1(x)}\right)^{q_2 + a_2} \dots \left(\frac{\Delta_r(x)}{\Delta_{r-1}(x)}\right)^{q_r + a_r} \frac{(\Delta_1(x) \dots \Delta_{r-1}(x))^{-\frac{1}{2}}}{\Delta_r(x)}.$$

It follows that

$$\Delta_{q+\rho+a-\frac{n}{r}}(x) = \frac{\prod_{i=1}^{r} \Delta_i(x)^{-\frac{1}{2}} \left(\frac{\Delta_i(x)}{\Delta_{i-1}(x)}\right)^{q_i + a_i}}{\sqrt{\det(x)}}. \tag{4.21}$$

On the other hand, using (3.13) we can write

$$\Gamma_\Omega(q + \rho + a) = (2\pi)^{\frac{r(r-1)}{4}} \prod_{i=1}^{r} \Gamma\left(q_i + a_i\right). \tag{4.22}$$

Inserting (4.22) and (4.21) in (4.20), we obtain

$$h_a(x) = \frac{e^{-tr(x)}}{(2\pi)^{\frac{r(r-1)}{4}} \sqrt{\det(x)}} \prod_{i=1}^{r} \left(\Delta_i(x)^{-\frac{1}{2}} e^{-\lambda_i} \sum_{q_i \in \mathbb{N}} \frac{\lambda_i^{q_i}}{q_i!\; \Gamma(q_i + a_i)} \left(\frac{\Delta_i(x)}{\Delta_{i-1}(x)}\right)^{q_i + a_i}\right).$$

$$= \frac{e^{-tr(x)}}{(2\pi)^{\frac{r(r-1)}{4}}\sqrt{\det(x)}} \prod_{i=1}^{r} \left(\Delta_i(x)^{-\frac{1}{2}} e^{-\lambda_i} \sum_{q_i \in \mathbb{N}} \frac{\lambda_i^{q_i}}{q_i!\ \Gamma(q_i+a_i)}\ \Delta_{e_i}(x)^{q_i+a_i} \right).$$

$$= \frac{e^{-tr(x)}}{(2\pi)^{\frac{r(r-1)}{4}}\sqrt{\det(x)}} \prod_{i=1}^{r} \left(\Delta_i(x)^{-\frac{1}{2}} e^{-\lambda_i} \Delta_{e_i}(x)^{a_i} \sum_{q_i \in \mathbb{N}} \frac{(\lambda_i \Delta_{e_i}(x))^{q_i}}{q_i!\ \Gamma(q_i+a_i)} \right).$$

Therefore if for $t > 0$ and $b > 0$, we define

$$g_b(t) = \sum_{k \in \mathbb{N}} \frac{t^k}{k!\ \Gamma(k+b)},$$

then we obtain

$$h_a(x) = \frac{e^{-tr(x)}}{(2\pi)^{\frac{r(r-1)}{4}}\sqrt{\det(x)}} \prod_{i=1}^{r} \left(\frac{e^{-\lambda_i}}{\sqrt{\Delta_i(x)}}\ \Delta_{e_i}(x)^{a_i} g_{a_i}(\lambda_i \Delta_{e_i}(x)) \right),$$

We now use the fact that for $t > 0$ and $b > 0$,, we have $\sum_{k \in \mathbb{N}} \frac{t^k}{k!\ \Gamma(k+b)} = t^{\frac{1-b}{2}}\ I(b-1, 2\sqrt{t})$, (see Abramowitz and Stegum (1974), page 375) to get

$$h_a(x) = \frac{e^{-tr(x)}}{(2\pi)^{\frac{r(r-1)}{4}}\sqrt{\det(x)}} \prod_{i=1}^{r} \left(\frac{\lambda_i^{\frac{1-a_i}{2}} e^{-\lambda_i}}{\sqrt{\Delta_i(x)}}\ \Delta_{e_i}(x)^{\frac{1+a_i}{2}}\ I\left(a_i - 1, 2\sqrt{\lambda_i \Delta_{e_i}(x)}\right) \right).$$

As $\mu(dx) = \lim_{a \longrightarrow 0} \mu_a(dx) = \left(\lim_{a \longrightarrow 0} h_a(x) \right) \mathbf{1}_\Omega(x)(dx)$, we deduce that

$$\mu(dx) = \frac{e^{-tr(x)}}{(2\pi)^{\frac{r(r-1)}{4}}\sqrt{\det(x)}} \prod_{i=1}^{r} \left(\frac{\sqrt{\lambda_i}\ e^{-\lambda_i}}{\sqrt{\Delta_{i-1}(x)}}\ I\left(-1, 2\sqrt{\lambda_i \Delta_{e_i}(x)}\right) \right) \mathbf{1}_\Omega(x)(dx).$$

To finish the proof of Theorem 4.1, we mention that for all $t > 0$, we have $I(1,t) = I(-1,t)$, (see Lebedev (1972), page 110). □

Next, we give the Laplace transform of μ. We denote $\kappa_r = \sum_{j=1}^{r-1} \frac{j}{2}\ e_j$, with $\kappa_1 = 0$.

Theorem 4.2 *For all $\theta \in I_r - \Omega$, we have*

$$L_\mu(\theta) = \Delta^*_{\kappa_r}(I_r - \theta) \exp\left(\sum_{i=1}^{r} \lambda_i \left(\Delta^*_{-e_{r-i+1}}(I_r - \theta) - 1 \right) \right).$$

Proof. Let X_k be a random variable with distribution $\widetilde{R}_k$, where $\widetilde{R}_k$ given by (4.19). Suppose that k follows the Poisson distribution defined in (4.17). Then, according to (3.14), the Laplace transform of the mixture μ of $\widetilde{R}_k$ by ν is given for $\theta \in I_r - \Omega$, by

$$L_\mu(\theta) = E\left(e^{\langle \theta, X_k \rangle} \right) = E\left(E\left(e^{\langle \theta, X_k \rangle} \mid k \right) \right) = E\left(\Delta_{k+\rho}((I_r - \theta)^{-1}) \right).$$

Using (3.6) and (3.7), we can write

$L_\mu(\theta) = E\left(\Delta^*_{-(k+\rho)^*}(I_r - \theta) \right).$

$$= E\left((\Delta_1^*(I_r-\theta))^{k_{r-1}-k_r-\frac{1}{2}}\dots(\Delta_{r-1}^*(I_r-\theta))^{k_1-k_2-\frac{1}{2}}(\Delta_r^*(I_r-\theta))^{-k_1}\right).$$
$$= \prod_{i=1}^{r-1}\left(\frac{1}{\Delta_i^*(I_r-\theta)}\right)^{\frac{1}{2}} E\left(\prod_{i=1}^{r}\left(\frac{\Delta_{i-1}^*(I_r-\theta)}{\Delta_i^*(I_r-\theta)}\right)^{k_{r-i+1}}\right).$$
$$= \prod_{i=1}^{r-1}\left(\frac{1}{\Delta_i^*(I_r-\theta)}\right)^{\frac{1}{2}} E\left(\prod_{i=1}^{r}\left(\Delta_{-e_i}^*(I_r-\theta)\right)^{k_{r-i+1}}\right).$$

It follows that

$$L_\mu(\theta) = \prod_{i=1}^{r-1}\left(\frac{1}{\Delta_i^*(I_r-\theta)}\right)^{\frac{1}{2}} E\left(\prod_{i=1}^{r} e^{k_{r-i+1}\log(\Delta_{-e_i}^*(I_r-\theta))}\right) \text{ for all } \theta \in I_r-\Omega.$$

Setting $\alpha(\theta) = \left(\log(\Delta_{-e_r}^*(I_r-\theta)), \log(\Delta_{-e_{r-1}}^*(I_r-\theta)), \dots, \log(\Delta_{-e_1}^*(I_r-\theta))\right)$, we get

$$L_\mu(\theta) = \prod_{i=1}^{r-1}\left(\frac{1}{\Delta_i^*(I_r-\theta)}\right)^{\frac{1}{2}} E\left(e^{\langle\alpha(\theta),k\rangle}\right) = \prod_{i=1}^{r-1}\left(\frac{1}{\Delta_i^*(I_r-\theta)}\right)^{\frac{1}{2}} L_\nu\left(\alpha(\theta)\right).$$

According to (4.18), we obtain for all $\theta \in I_r-\Omega$,

$$L_\mu(\theta) = \prod_{i=1}^{r-1}\left(\frac{1}{\Delta_i^*(I_r-\theta)}\right)^{\frac{1}{2}} \prod_{i=1}^{r} e^{\lambda_i\left(\left(\Delta_{-e_{r-i+1}}^*(I_r-\theta)\right)-1\right)}. \tag{4.23}$$

Using (3.6), we have that $\prod_{i=1}^{r-1}\left(\frac{1}{\Delta_i^*(I_r-\theta)}\right)^{\frac{1}{2}} = \Delta_{\kappa_r}^*(I_r-\theta)$. Inserting this in (4.23), we get the result. □

The following theorem gives the domain of the means and the variance function of the NEF generated by the mixture μ.

Theorem 4.3
i) The domain of the means of the NEF $F = F(\mu)$ is Ω.
ii) The variance function of F evaluated for $m \in \Omega$ is equal to

$$V_F(m) = -\frac{1}{2}P\left(\sum_{i=1}^{r}\frac{1}{b_i(m)}\left(\left(P_{r-i+1}^*(m^{-1})\right)^{-1} - \left(P_{r-i}^*(m^{-1})\right)^{-1}\right)\right)$$
$$+\sum_{i=1}^{r}(b_{r-i+1}(m)-b_{r-i}(m))\left[P\left(\sum_{j=1}^{i}\frac{1}{b_{r-j+1}(m)}\left(\left(P_j^*(m^{-1})\right)^{-1} - \left(P_{j-1}^*(m^{-1})\right)^{-1}\right)\right)\right]$$
$$+\sum_{i=1}^{r}\left(\frac{1}{b_{r-i+1}(m)} - \frac{r-i}{2(b_{r-i+1}(m))^2}\right)\left[\left(\left(P_i^*(m^{-1})\right)^{-1} - \left(P_{i-1}^*(m^{-1})\right)^{-1}\right)\right.$$
$$\left.\otimes\left(\left(P_i^*(m^{-1})\right)^{-1} - \left(P_{i-1}^*(m^{-1})\right)^{-1}\right)\right], \tag{4.24}$$

where $b_0(m) = -\frac{1}{2}$ and for all $i \in \{1,\dots,r\}$,

$$b_i(m) = \frac{i-1}{4} + \sqrt{\left(\frac{i-1}{4}\right)^2 + \lambda_i\Delta_{e_i}(m)}. \tag{4.25}$$

Before embarking in the proof of the theorem, we mention that usually for the calculation of the variance function, we set $m = k'_\mu(\theta)$ and we determine its reciprocal $\theta = \psi_\mu(m)$. This is difficult to do in the present situation, however, we will next show that without getting the explicit expression of $\psi_\mu(m)$, we are able to express $(P_i^*(I_r - \psi_\mu(m)))^{-1}$ and $\Delta^*_{-e_i}(I_r - \psi_\mu(m))$ in terms of $b_i(m)$ defined in (4.25). This is crucial for the calculation of the variance function.

Proposition 4.4 *For all $i \in \{1, \ldots, r\}$,*

i) $(P_i^*(I_r - \psi_\mu(m)))^{-1} = \sum_{j=r-i+1}^{r} \frac{1}{b_j(m)} \left[\left(P^*_{r-j+1}(m^{-1})\right)^{-1} - \left(P^*_{r-j}(m^{-1})\right)^{-1} \right].$

$$\text{ii) } \Delta^*_{-e_i}(I_r - \psi_\mu(m)) = \frac{b_{r-i+1}(m)}{\lambda_{r-i+1}} - \frac{r-i}{2\lambda_{r-i+1}}. \tag{4.26}$$

Proof. We have that that for all $\theta \in \Theta(\mu) = I_r - \Omega$,

$$k_\mu(\theta) = \sum_{i=1}^{r} \lambda_i \left(\left(\Delta^*_{-e_{r-i+1}}(I_r - \theta) \right) - 1 \right) + \log(\Delta^*_{\kappa_r}(I_r - \theta)). \tag{4.27}$$

As for all $i \in \{1, \ldots, r\}$, the map $\varphi_i: \ x \longmapsto \log \Delta_i^*(x)$ is differentiable on the cone Ω and $\varphi_i'(x) = (P_i^*(x))^{-1}$, then

$$\left(\Delta^*_{-e_i}(x)\right)' = \left(\frac{\Delta^*_{i-1}(x)}{\Delta^*_i(x)}\right)' = \Delta^*_{-e_i}(x) \left((P^*_{i-1}(x))^{-1} - (P_i^*(x))^{-1} \right), \tag{4.28}$$

and for $r \geq 2$, we have

$$(\log(\Delta^*_{\kappa_r}(x)))' = -\frac{1}{2} \sum_{i=1}^{r-1} (P_i^*(x))^{-1} \tag{4.29}$$

Differentiating (4.27) and taking into account (4.28) and (4.29), we get

$$k'_\mu(\theta) = \sum_{i=1}^{r} \left(\lambda_{r-i+1} \Delta^*_{-e_i}(I_r - \theta) - \lambda_{r-i} \Delta^*_{-e_{i+1}}(I_r - \theta) + \frac{1}{2} \right) (P_i^*(I_r - \theta))^{-1} - \frac{1}{2} (P_r^*(I_r - \theta))^{-1}, \tag{4.30}$$

where $\lambda_0 = 0$.

Let $\theta \in I_r - \Omega$, and let u be the unique element of $\mathcal{T}_l^+$ such that $I_r - \theta = u^{*-1}(I_r)$. Then, for all $i \in \{1, \ldots, r\}$, we have

$$(P_i^*(I_r - \theta))^{-1} = \left(P_i^*(u^{*-1}(I_r))\right)^{-1} = \left(P_i^*\left((u(I_r))^{-1}\right)\right)^{-1}.$$

This according to (3.9) implies that, for all $i \in \{1, \ldots, r\}$,

$$(P_i^*(I_r - \theta))^{-1} = u\left(\sum_{j=r-i+1}^{r} c_j\right). \tag{4.31}$$

On the other hand, using (3.10), we can write for all $i \in \{1, \ldots, r\}$,

$$\Delta^*_{-e_i}(I_r - \theta) = \Delta^*_{-e_i}\left(u^{*-1}(I_r)\right) = \Delta_{e_i^*}(u(I_r)) = \Delta_{e_{r-i+1}}(u(I_r)) = \frac{\Delta_{r-i+1}(u(I_r))}{\Delta_{r-i}(u(I_r))}.$$

This with (3.11) imply that for all $i \in \{1, \dots, r\}$,

$$\Delta^*_{-e_i} = u^2_{r-i+1}, \tag{4.32}$$

where the u_i are defined in (3.12).
Using (4.31) and (4.32), we deduce from (4.30) that

$$k'_\mu(\theta) = \sum_{i=1}^{r} \left(\lambda_{r-i+1} u^2_{r-i+1} - \lambda_{r-i} u^2_{r-i} + \frac{1}{2}\right) u \left(\sum_{j=r-i+1}^{r} c_j\right) - \frac{1}{2}\, u \left(\sum_{i=1}^{r} c_i\right).$$

This after a standard calculation, gives

$$k'_\mu(\theta) = \sum_{i=1}^{r} \left(\lambda_i u_i^2 + \frac{i-1}{2}\right) u(c_i) = u\left(\sum_{i=1}^{r} a_i(\theta) c_i\right), \tag{4.33}$$

where for all $i \in \{1, \dots, r\}$,

$$a_i(\theta) = \frac{i-1}{2} + \lambda_i u_i^2 > 0. \tag{4.34}$$

Let now $m = k'_\mu(\theta)$ be an element of Ω, then using the Cholesky decomposition, there exists a unique $v \in \mathcal{T}_l^+$ such that $m = v(I_r)$. According to (4.33), we have

$$v(I_r) = m = k'_\mu(\theta) = u\left(\sum_{i=1}^{r} a_i(\psi_\mu(m)) c_i\right) = u\left(P\left(\sum_{i=1}^{r} \sqrt{a_i(\psi_\mu(m))}\; c_i\right)\left(\sum_{i=1}^{r} c_i\right)\right).$$

Therefore

$$v(I_r) = u\left(P\left(\sum_{i=1}^{r} \sqrt{a_i(\psi_\mu(m))}\; c_i\right)(I_r)\right).$$

It follows that

$$u = v \sum_{i=1}^{r} \frac{1}{\sqrt{a_i(\psi_\mu(m))}}\, c_i. \tag{4.35}$$

Using (4.34), (3.11) and (4.35), we deduce that

$$\begin{aligned} a_i(\psi_\mu(m)) &= \frac{i-1}{2} + \lambda_i \Delta_{e_i}\left(\left(v \sum_{j=1}^{r} \frac{1}{\sqrt{a_j(\psi_\mu(m))}}\; c_j\right)\left(\sum_{j=1}^{r} \frac{1}{\sqrt{a_j(\psi_\mu(m))}}\; c_j\; v^*\right)\right). \\ &= \frac{i-1}{2} + \lambda_i \Delta_{e_i}\left(v \sum_{j=1}^{r} \frac{1}{a_j(\psi_\mu(m))}\; c_j\; v^*\right). \end{aligned} \tag{4.36}$$

Inserting (3.8) in (4.36), we obtain that for all $i \in \{1, \dots, r\}$,

$$a_i(\psi_\mu(m)) = \frac{i-1}{2} + \lambda_i \Delta_{e_i}\left(v\left(\sum_{j=1}^{r} \frac{1}{a_j(\psi_\mu(m))}\; c_j\right)\right).$$

Hence $a_i(\psi_\mu(m))$ satisfies the equation

$$a_i(\psi_\mu(m)) = \frac{i-1}{2} + \frac{\lambda_i v_i^2}{a_i(\psi_\mu(m))},$$

where v_i is defined in (3.12).

As $a_i(\psi_\mu(m)) > 0$, then for all $i \in \{1, \ldots, r\}$,

$$a_i(\psi_\mu(m)) = \frac{i-1}{4} + \sqrt{\left(\frac{i-1}{4}\right)^2 + \lambda_i v_i^2}.$$

On the other hand, since $m = v(I_r)$, then using (3.11), we have that $v_i^2 = \Delta_{e_i}(m)$.

Setting

$$b_i(m) = a_i(\psi_\mu(m)), \tag{4.37}$$

we deduce that for all $i \in \{1, \ldots, r\}$,

$$b_i(m) = \frac{i-1}{4} + \sqrt{\left(\frac{i-1}{4}\right)^2 + \lambda_i \Delta_{e_i}(m)}.$$

$i)$ With the notations used above, we can write for all $i \in \{1, \ldots, r\}$,

$$\begin{aligned}
(P_i^*(I_r - \psi_\mu(m)))^{-1} &= (P_i^*(u^{*-1}(I_r)))^{-1} = u\left(\sum_{j=r-i+1}^{r} c_j\right). \\
&= \left(v\sum_{i=1}^{r}\frac{1}{\sqrt{b_i(m)}}\, c_i\right)\left(\sum_{j=r-i+1}^{r} c_j\right)\left(\sum_{i=1}^{r}\frac{1}{\sqrt{b_i(m)}}\, c_i\ v^*\right). \\
&= v\sum_{j=r-i+1}^{r}\frac{1}{b_j(m)}\, c_j\ v^* = v\left(\sum_{j=r-i+1}^{r}\frac{1}{b_j(m)}\, c_j\right).
\end{aligned}$$

It follows that for all $i \in \{1, \ldots, r\}$,

$$(P_i^*(I_r - \psi_\mu(m)))^{-1} = \sum_{j=r-i+1}^{r}\frac{1}{b_j(m)}\, v(c_j). \tag{4.38}$$

As $m = v(I_r)$, then according to (3.9) we have for all $j \in \{1, \ldots, r\}$,

$$v(c_j) = v\left(\sum_{i=j}^{r} c_i - \sum_{i=j+1}^{r} c_i\right) = \left(P_{r-j+1}^*(m^{-1})\right)^{-1} - \left(P_{r-j}^*(m^{-1})\right)^{-1}.$$

Inserting this in (4.38), we deduce that

$$(P_i^*(I_r - \psi_\mu(m)))^{-1} = \sum_{j=r-i+1}^{r}\frac{1}{b_j(m)}\left[\left(P_{r-j+1}^*(m^{-1})\right)^{-1} - \left(P_{r-j}^*(m^{-1})\right)^{-1}\right]. \tag{4.39}$$

Consequently

$$(P_i^*(I_r - \psi_\mu(m)))^{-1} - (P_{i-1}^*(I_r - \psi_\mu(m)))^{-1} = \frac{1}{b_{r-i+1}(m)}\left(\left(P_i^*(m^{-1})\right)^{-1} - \left(P_{i-1}^*(m^{-1})\right)^{-1}\right). \tag{4.40}$$

$ii)$ According to (3.6) we have that

$$\Delta_i^*(I_r - \psi_\mu(m)) = \Delta_i^*(u^{*-1}(I_r)) = \Delta_{\vartheta_i}^*(u^{*-1}(I_r)),$$

where $\vartheta_i = \sum_{j=1}^{i} e_j$. Using (3.10), (4.35) and (4.37), we deduce that

$$\Delta_i^*(I_r - \psi_\mu(m)) = \Delta_{-\vartheta_i^*}(u(I_r)) = \Delta_{-\vartheta_i^*}(uu^*).$$

$$= \Delta_{-\vartheta_i^*}\left(\left(v\sum_{j=1}^{r}\frac{1}{\sqrt{b_j(m)}}\ c_j\right)\left(\sum_{j=1}^{r}\frac{1}{\sqrt{b_j(m)}}\ c_j\ v^*\right)\right).$$

$$= \Delta_{-\vartheta_i^*}\left(v\sum_{j=1}^{r}\frac{1}{b_j(m)}\ c_j\ v^*\right) = \Delta_{-\vartheta_i^*}\left(v\left(\sum_{j=1}^{r}\frac{1}{b_j(m)}\ c_j\right)\right).$$

Thus, using (3.5), we obtain

$$\Delta_i^*(I_r - \psi_\mu(m)) = \frac{\Delta_{r-i}\left(v\left(\sum_{j=1}^{r}\frac{1}{b_j(m)}\ c_j\right)\right)}{\Delta_r\left(v\left(\sum_{j=1}^{r}\frac{1}{b_j(m)}\ c_j\right)\right)}.$$

It follows that for all $i \in \{1, \ldots, r\}$,

$$\Delta_{-e_i}^*(I_r - \psi_\mu(m)) = \frac{\Delta_{i-1}^*(I_r - \psi_\mu(m))}{\Delta_i^*(I_r - \psi_\mu(m))} = \frac{\Delta_{r-i+1}\left(v\left(\sum_{j=1}^{r}\frac{1}{b_j(m)}\ c_j\right)\right)}{\Delta_{r-i}\left(v\left(\sum_{j=1}^{r}\frac{1}{b_j(m)}\ c_j\right)\right)}.$$

Using (3.11), we deduce that $\Delta_{-e_i}^*(I_r - \psi_\mu(m)) = \dfrac{v_{r-i+1}^2}{b_{r-i+1}(m)}$.

As $v_{r-i+1}^2 = \dfrac{\Delta_{r-i+1}(v(I_r))}{\Delta_{r-i}(v(I_r))} = \dfrac{\Delta_{r-i+1}(m)}{\Delta_{r-i}(m)}$, we get $\Delta_{-e_i}^*(I_r - \psi_\mu(m)) = \dfrac{\Delta_{r-i+1}(m)}{b_{r-i+1}(m)\Delta_{r-i}(m)}$.

This with (3.5) imply that

$$\Delta_{-e_i}^*(I_r - \psi_\mu(m)) = \frac{\Delta_{e_{r-i+1}(m)}}{b_{r-i+1}(m)}. \tag{4.41}$$

According to (4.25), we have for all $i \in \{1, \ldots, r\}$, $\Delta_{e_i}(m) = \dfrac{(b_i(m))^2 - \frac{i-1}{2}b_i(m)}{\lambda_i}$.

Inserting this in (4.41), we obtain

$$\Delta_{-e_i}^*(I_r - \psi_\mu(m)) = \frac{(b_{r-i+1}(m))^2 - \frac{r-i}{2}b_{r-i+1}(m)}{\lambda_{r-i+1}b_{r-i+1}(m)} = \frac{b_{r-i+1}(m)}{\lambda_{r-i+1}} - \frac{r-i}{2\lambda_{r-i+1}}. \qquad \square$$

We come now to the proof of Theorem 4.3.

Proof of Theorem 4.3

i) As the a_i defined in (4.34) are strictly positive, then using (4.33) we deduce that

$$M_F = k'_\mu(\Theta(\mu)) = k'_\mu(I_r - \Omega) \subseteq \Omega. \tag{4.42}$$

Conversely, consider $y \in \Omega$, then using the Cholesky decomposition, there exists a unique $w \in \mathcal{T}_l^+$ such that

$$y = w(I_r) = w\left(\sum_{i=1}^{r} c_i\right) = w\left(P\left(\sum_{i=1}^{r}\frac{1}{\sqrt{a_i(\theta)}}c_i\right)\left(\sum_{i=1}^{r} a_i(\theta)c_i\right)\right),$$

where the $a_i(\theta)$ are given by (4.34).

Let $\theta = I_r - u^{*-1}(I_r)$, where $u = w\sum_{i=1}^{r}\frac{1}{\sqrt{a_i(\theta)}}\, c_i$. Then $y = u\left(\sum_{i=1}^{r} a_i(\theta)c_i\right)$.

This, using (4.33), gives $y = k'_\mu(\theta) \in k'_\mu(I_r - \Omega) = k'_\mu(\Theta(\mu))$. Hence $\Omega \subseteq k'_\mu(\Theta(\mu))$, and according to (4.42), we obtain $M_F = k'_\mu(\Theta(\mu)) = \Omega$.

ii) Differentiating (4.30) and using (4.28) and the fact that for all $i \in \{1,\ldots,r\}$ and $x \in \Omega$,

$$\left((P_i^*(x))^{-1}\right)' = -P\left((P_i^*(x))^{-1}\right),$$

where P is defined in (3.4), we get for all $\theta \in I_r - \Omega$,

$$\begin{aligned}
k''_\mu(\theta) = &-\frac{1}{2}P\left((P_r^*(I_r-\theta))^{-1}\right)\\
&+\sum_{i=1}^{r}\left(\lambda_{r-i+1}\Delta^*_{-e_i}(I_r-\theta) - \lambda_{r-i}\Delta^*_{-e_{i+1}}(I_r-\theta)+\frac{1}{2}\right)\left(P\left((P_i^*(I_r-\theta))^{-1}\right)\right)\\
&+\sum_{i=1}^{r}\lambda_{r-i+1}\Delta^*_{-e_i}(I_r-\theta)\left((P_i^*(I_r-\theta))^{-1} - (P_{i-1}^*(I_r-\theta))^{-1}\right)\otimes(P_i^*(I_r-\theta))^{-1}\\
&-\sum_{i=1}^{r}\lambda_{r-i}\Delta^*_{-e_{i+1}}(I_r-\theta)\left((P_{i+1}^*(I_r-\theta))^{-1} - (P_i^*(I_r-\theta))^{-1}\right)\otimes(P_i^*(I_r-\theta))^{-1}.
\end{aligned}$$

It follows that for all $\theta \in I_r - \Omega$,

$$\begin{aligned}
k''_\mu(\theta) = &-\frac{1}{2}P\left((P_r^*(I_r-\theta))^{-1}\right)\\
&+\sum_{i=1}^{r}\left(\lambda_{r-i+1}\Delta^*_{-e_i}(I_r-\theta) - \lambda_{r-i}\Delta^*_{-e_{i+1}}(I_r-\theta)+\frac{1}{2}\right)\left(P\left((P_i^*(I_r-\theta))^{-1}\right)\right)\\
&+\sum_{i=1}^{r}\lambda_{r-i+1}\Delta^*_{-e_i}(I_r-\theta)\\
&\times\left[\left((P_i^*(I_r-\theta))^{-1} - (P_{i-1}^*(I_r-\theta))^{-1}\right)\otimes\left((P_i^*(I_r-\theta))^{-1} - (P_{i-1}^*(I_r-\theta))^{-1}\right)\right].
\end{aligned}$$

We need only to replace θ by $\psi_\mu(m)$, then insert (4.26), (4.39) and (4.40) to get the expression of the variance function of $F = F(\mu)$ given in (4.24). □

To close the paper, we mention that particular statements of these theorems may be given for the case where the parameter $s = (s_1, s_2, \ldots, s_r)$ in the Riesz distribution is such that $s_1 = \ldots = s_r = p$ which corresponds to the Wishart distribution. In this case, the distributions in the model are the $W(p + \frac{r-1}{2}, I_r)$ (see (3.15)), and the mixing parameter p has the Poisson distribution $\nu(dx) = \sum_{q\in\mathbb{N}}\frac{\lambda^q e^{-\lambda}}{q!}\delta_q(dx)$.

The mixture of $W(p + \frac{r-1}{2}, I_r)$ with respect to ν is then

$$\mu(dx) = \frac{e^{-\lambda} - tr(x)}{(2\pi)^{\frac{r(r-1)}{4}}\det(x)}\prod_{i=1}^{r}(\lambda\det(x))^{\frac{1-\frac{r-i}{2}}{2}}\, I\left(\frac{r-i}{2}-1, 2\sqrt{\lambda\det(x)}\right)\mathbf{1}_\Omega(x)(dx),$$

where $I(b,t)$ is the modified Bessel function of the first kind of order b.
The cumulant function of the μ is given for $\theta \in I_r - \Omega$, by

$$k_\mu(\theta) = \frac{1-r}{2}\log\left(\det\left(I_r - \theta\right)\right) + \frac{\lambda}{\det\left(I_r - \theta\right)} - \lambda,$$

and the variance function of the generated NEF is defined on Ω by

$$V_F(m) = \left(\frac{r-1}{2} + \frac{\lambda\det(m)}{(b\,(m))^r}\right) P\left(\sum_{i=1}^{r}\frac{1}{b\,(m)}\left[\left(P_i^*\left(m^{-1}\right)\right)^{-1} - \left(P_{i-1}^*\left(m^{-1}\right)\right)^{-1}\right]\right)$$
$$+\frac{\lambda\det(m)}{(b\,(m))^{r+2}}\left[\sum_{i=1}^{r}\left(P_i^*\left(m^{-1}\right)\right)^{-1} - \left(P_{i-1}^*\left(m^{-1}\right)\right)^{-1}\right]\otimes\left[\sum_{i=1}^{r}\left(P_i^*\left(m^{-1}\right)\right)^{-1} - \left(P_{i-1}^*\left(m^{-1}\right)\right)^{-1}\right],$$

where $b(m)$ is the unique positive solution of the equation $b(m) = \dfrac{r-1}{2} + \dfrac{\lambda\det(m)}{(b(m)))^r}$.

In the particular case where $r = 1$, we obtain $V_F(m) = \dfrac{2m^{\frac{3}{2}}}{\sqrt{\lambda}}$, for all $m > 0$.

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